Scale-invariant boundary Harnack principle on inner uniform domains in fractal-type spaces

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Abstract

We prove a scale-invariant boundary Harnack principle (BHP) on inner uniform domains in metric measure spaces. We prove our result in the context of strictly local, regular, symmetric Dirichlet spaces, without assuming that the Dirichlet form induces a metric that generates the original topology on the metric space. Thus, we allow the underlying space to be fractal, e.g. the Sierpinski gasket.

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Introduction

Given the notion of harmonic functions, the boundary Harnack principle is a property of a domain that provides control over the ratio of two harmonic functions in that domain near some part of the boundary where the two functions satisfy the Dirichlet boundary condition. Whether a given domain satisfies the boundary Harnack principle depends on the geometry of its boundary. We are interested in the case when the domain is uniform or, more generally, inner uniform.

Aikawa and Ancona proved a scale-invariant boundary Harnack principle on uniform and inner uniform domains, respectively, in Euclidean space.

In [8], Gyrya and Saloff-Coste generalized Aikawa's approach to uniform domains in strictly local, regular, symmetric Dirichlet spaces of Harnack-type that admit a carré du champ. Moreover, they deduced that the boundary Harnack principle also holds on inner uniform domains, by considering the inner uniform domain as a uniform domain in a different metric space, namely the completion of the inner uniform domain with respect to its inner metric.

In [10], we proved the boundary Harnack principle directly on inner uniform domains, when the underlying space is a strictly local, regular (possibly non-symmetric) Dirichlet space of Harnack-type.

In this paper, we assume that the underlying space $(X, d, \mu, \mathcal{E}, D(\mathcal{E}))$ is a metric measure Dirichlet space in the sense of [5]. The parabolic Harnack inequality PHI(Ψ) which we assume to hold on the space, has a space-time scaling that is captured by the function

$$\Psi(s) = \Psi_{\beta,\bar{\beta}}(s) = \begin{cases} s^{\bar{\beta}}, & \text{if } s \le 1, \\ s^{\beta}, & \text{if } s > 1, \end{cases}$$

for some $\beta, \bar{\beta} \ge 2$. In the classical case, $\Psi(r) = r^2$. Different values of β and $\bar{\beta}$ allow for fractal-type spaces, such as the Sierpinski gasket. For example, removing the bottom line in the Sierpinski gasket produces a subset that is an inner uniform domain in the Sierpinski gasket.

In fractal-type spaces, the energy measure of the Dirichlet form can be singular with respect to the measure of the underlying metric measure space. We thus do not make any assumptions on the pseudo-metric induced by the Dirichlet form, in fact we do not consider it at all. This is contrary to [8] and [10].

The main result of this paper is the following scale-invariant boundary Harnack principle.

Theorem 0.1 *Let Ω be an inner uniform domain in the metric measure space X. Suppose that X is equipped with a strictly local, regular, symmetric Dirichlet form that satisfies the parabolice Harnack inequality* PHI(Ψ)*. Then there exist constants $A_0, A_1 \in (1, \infty)$ such that for any $\xi \in \partial\Omega$, $R = R(\xi, \Omega) > 0$ small, and any two non-negative weak solutions u and v of the Laplace equation $Lu = 0$ in $B_\Omega(\xi, A_0 r)$ with weak Dirichlet boundary condition along $\partial\Omega$, we have*

$$\frac{u(x)}{u(x')} \le A_1 \frac{v(x)}{v(x')},$$

for all $x, x' \in B_\Omega(\xi, r)$. Here $B_\Omega(\xi, r) = \{z \in \Omega : d_\Omega(\xi, z) < r\}$ and d_Ω is the intrinsic distance of the domain. The constant A_1 depends only on the Harnack constant H_0, on β, $\bar{\beta}$, and on the inner-uniformity constants c_u, C_u.

In Section 1, we review some general properties of Dirichlet spaces and local weak solutions. In Section 2 we describe the parabolic Harnack inequality and equivalent conditions that we impose on the space. In Section 3 we prove estimates for the Dirichlet heat kernel on balls. After recalling the definition and some properties of inner uniform domains, we give estimates for Green's functions on balls intersected with an inner uniform domain. In Section 4 we give a proof of the boundary Harnack principle.

1 Local weak solutions and capacity

1.1 Local weak solutions

Let $(X, d, \mu, \mathcal{E}, D(\mathcal{E}))$ be a metric measure Dirichlet space (MMD space). That is, X is a connected, locally compact, separable, complete metric space and the metric d is geodesic. μ is a non-negative Borel measure on X that is finite on compact sets and positive on non-empty open sets. $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric, strictly local, regular Dirichlet form on $L^2(X, \mu)$. See [7]. The semigroup $(P_t)_{t\geq 0}$ associated with $(\mathcal{E}, D(\mathcal{E}))$ is assumed to be conservative. We denote by $(L, D(L))$ the infinitesimal generator of $(\mathcal{E}, D(\mathcal{E}))$.

There exists a measure-valued quadratic form $d\Gamma$ defined by

$$\int f\, d\Gamma(u, u) = \mathcal{E}(uf, u) - \frac{1}{2}\mathcal{E}(f, u^2), \quad \forall f, u \in D(\mathcal{E}) \cap L^\infty(X),$$

and extended to unbounded functions by setting $\Gamma(u, u) = \lim_{n\to\infty} \Gamma(u_n, u_n)$, where $u_n = \max\{\min\{u, n\}, -n\}$. Using polarization, we obtain a bilinear form $d\Gamma$. In particular,

$$\mathcal{E}(u, v) = \int d\Gamma(u, v), \quad \forall u, v \in D(\mathcal{E}).$$

For $U \subset X$ open, set

$$\mathcal{F}_{\text{loc}}(U) = \{f \in L^2_{\text{loc}}(U) : \forall \text{ compact } K \subset U,\ \exists f^\sharp \in D(\mathcal{E}), f = f^\sharp\big|_K \text{ a.e.}\},$$

where $L^2_{\text{loc}}(U)$ is the space of functions that are locally in $L^2(U)$. For $f, g \in \mathcal{F}_{\text{loc}}(U)$ we define $\Gamma(f, g)$ locally by $\Gamma(f, g)\big|_K = \Gamma(f^\sharp, g^\sharp)\big|_K$, where $K \subset U$ is compact and $f^\sharp, g^\sharp$ are functions in $D(\mathcal{E})$ such that $f = f^\sharp$, $g = g^\sharp$ a.e. on K. Furthermore, set

$$\mathcal{F}(U) = \{u \in \mathcal{F}_{\text{loc}}(U) : \int_U |u|^2 d\mu + \int_U d\Gamma(u, u) < \infty\},$$
$$\mathcal{F}_{\text{c}}(U) = \{u \in \mathcal{F}(U) : \text{ The essential support of } u \text{ is compact in } U\}.$$

Definition 1.1 *For an open set $U \subset X$, the Dirichlet-type form on U is defined as*

$$\mathcal{E}^D_U(f, g) = \mathcal{E}(f, g), \quad f, g \in D(\mathcal{E}^D_U),$$

where the domain $D(\mathcal{E}^D_U) = \mathcal{F}^0(U)$ is defined as the closure of the space $C_0^\infty(U)$ of all smooth functions with compact support in U. The closure is taken in the norm $\mathcal{E}^D_{U,1}(f, f)^{\frac{1}{2}} = (\mathcal{E}^D_U(f, f) + \|f\|_2)^{\frac{1}{2}}$.

The Dirichlet form $(\mathcal{E}^D_U, D(\mathcal{E}^D_U))$ is associated with a semigroup $P^D_U(t)$, $t > 0$. Using the reasoning in [14][Section 2.4], one can show that under the hypothesis that the underlying space satisfies the parabolic Harnack inequality PHI(Ψ), the semigroup has a continuous kernel $p^D_U(t, x, y)$. Moreover, the map $y \mapsto p^D_U(t, x, y)$ is in $\mathcal{F}^0(U)$.

Definition 1.2 *Let $V \subset U$ be open. Set*

$$\begin{aligned}\mathcal{F}^0_{\mathrm{loc}}(U,V) = &\{f \in L^2_{\mathrm{loc}}(V,\mu) : \forall \text{ open } A \subset V \text{ rel. compact in } \overline{U} \text{ with} \\ & d_U(A, U \setminus V) > 0, \exists f^\sharp \in \mathcal{F}^0(U) : f^\sharp = f\ \mu\text{-a.e. on } A\},\end{aligned}$$

where

$$d_U(A, U\backslash V) = \inf\big\{\mathrm{length}(\gamma)\big|\gamma : [0,1] \to U \text{ continuous}, \gamma(0) \in A, \gamma(1) \in U\backslash V\big\}.$$

Definition 1.3 *Let $V \subset U$ be open and $f \in \mathcal{F}_{\mathrm{c}}(V)'$, the dual space of $\mathcal{F}_{\mathrm{c}}(V)$ (identify $L^2(X,\mu)$ with its dual space using the scalar product). A function $u : V \to \mathbb{R}$ is a* local weak solution *of the Laplace equation $-Lu = f$ in V, if*

(i) $u \in \mathcal{F}_{\mathrm{loc}}(V)$,

(ii) For any function $\phi \in \mathcal{F}_{\mathrm{c}}(V)$, $\mathcal{E}(u,\phi) = \int f\phi\, d\mu$.

If in addition

$$u \in \mathcal{F}^0_{\mathrm{loc}}(U,V),$$

then u is a local weak solution with Dirichlet boundary condition *along ∂U.*

For a time interval I and a Hilbert space H, let $L^2(I \to H)$ be the Hilbert space of those functions $v : I \to H$ such that

$$\|v\|_{L^2(I\to H)} = \left(\int_I \|v(t)\|_H\, dt\right)^{1/2} < \infty.$$

Let $W^1(I \to H) \subset L^2(I \to H)$ be the Hilbert space of those functions $v : I \to H$ in $L^2(I \to H)$ whose distributional time derivative v' can be represented by functions in $L^2(I \to H)$, equipped with the norm

$$\|v\|_{W^1(I\to H)} = \left(\int_I \|v(t)\|_H + \|v'(t)\|_H\, dt\right)^{1/2} < \infty.$$

Let

$$\mathcal{F}(I \times X) = L^2(I \to D(\mathcal{E})) \cap W^1(I \to D(\mathcal{E})'), \tag{1}$$

where $D(\mathcal{E})'$ denotes the dual space of $D(\mathcal{E})$. Similarly, define $\mathcal{F}^0(I \times U)$ by replacing $D(\mathcal{E})$ by $\mathcal{F}^0(U)$ in (1). Let

$$\mathcal{F}_{\mathrm{loc}}(I \times U)$$

be the set of all functions $u : I \times U \to \mathbb{R}$ such that for any open interval J that is relatively compact in I, and any open subset A relatively compact in U, there exists a function $u^\sharp \in \mathcal{F}(I \times U)$ such that $u^\sharp = u$ a.e. in $J \times A$. Let

$$\mathcal{F}_{\mathrm{c}}(I \times U) = \{u \in \mathcal{F}_{\mathrm{loc}}(I \times U) : u(t,\cdot) \text{ has compact support in } U \text{ for a.e. } t \in I\}.$$

For an open subset $V \subset U$, let $Q = I \times V$ and let

$$\mathcal{F}^0_{\text{loc}}(U, Q)$$

be the set of all functions $u : Q \to \mathbb{R}$ such that for any open interval J that is relatively compact in I, and any open set $A \subset V$ relatively compact in $\bar{U}$ with $d_U(A, U \setminus V) > 0$, there exists a function $u^\sharp \in \mathcal{F}^0(I \times U)$ such that $u^\sharp = u$ a.e. in $J \times A$.

Definition 1.4 *Let I be an open interval and $V \subset U$ open. Set $Q = I \times V$. A function $u : Q \to \mathbb{R}$ is a* local weak solution *of the heat equation $(\partial_t - L)u = 0$ in Q, if*

(i) $u \in \mathcal{F}_{\text{loc}}(Q)$,

(ii) For any open interval J relatively compact in I,

$$\forall \phi \in \mathcal{F}_{\text{c}}(Q), \quad \int_J \int_V \partial_t \phi \, u \, d\mu \, dt - \int_J \mathcal{E}(\phi(t, \cdot), u(t, \cdot)) dt = 0.$$

If in addition

$$u \in \mathcal{F}^0_{\text{loc}}(U, Q),$$

then u is a local weak solution with Dirichlet boundary condition *along ∂U.*

1.2 Capacity

Let $U \subset X$ be open and relatively compact. The extended Dirichlet space $\mathcal{F}_e(U)$ is defined (see [7]) as the family of all measurable, almost everywhere finite functions u such that there exists an approximating sequence $u_n \in D(\mathcal{E}^D_U)$ that is $\mathcal{E}^D_U$-Cauchy and $u = \lim u_n$ almost everywhere.

For any set $A \subset U$ define

$$\mathcal{L}_{A,U} = \{u \in \mathcal{F}_e(U) : \tilde{u} \geq 1 \text{ q.e. on } A\},$$

where $\tilde{u}$ is a quasi-continuous modification of u. If $\mathcal{L}_{A,U} \neq \emptyset$, there exists a unique function $e_A \in \mathcal{L}_{A,U}$ such that $\mathcal{E}(e_A, w) \geq \mathcal{E}(e_A, e_A)$ holds for all $w \in \mathcal{L}_{A,U}$. The 0-*capacity* of A in U is defined by

$$\text{Cap}_U(A) = \begin{cases} \mathcal{E}(e_A, e_A), & \mathcal{L}_{A,U} \neq \emptyset \\ +\infty, & \mathcal{L}_{A,U} = \emptyset. \end{cases}$$

By [6][Proposition VI.4.3], $e_A = G_U \nu_A$, where ν_A is a finite measure with $\text{supp}(\nu_A)$ contained in the completion of A, and G_U is the Green's function associated with $\mathcal{E}^D_U$ (see [6][page 256]). Thus, if A is compact,

$$\text{Cap}_U(A) = \mathcal{E}(e_A, e_A) = \mathcal{E}(G_U \nu_A, e_A) = \int e_A \, d\nu_A = \nu_A(A).$$

2 Parabolic Harnack inequality and equivalent conditions

Let $(X, d, \mu, \mathcal{E}, D(\mathcal{E}))$ be a MMD space.

Definition 2.1 (X, μ) *satisfies the* volume doubling property VD *if there exists a constant* $D_0 \in (0, \infty)$ *such that for every* $x \in X$*,* $R > 0$*,*

$$V(x, 2R) \leq D_0 \, V(x, R),$$

where $V(x, R) = \mu(B(x, R))$ *denotes the volume of the ball* $B(x, R)$*.*

Let $\beta, \bar{\beta} \geq 2$ and set

$$\Psi(s) = \Psi_{\beta,\bar{\beta}}(s) = \begin{cases} s^{\bar{\beta}}, & \text{if } s \leq 1, \\ s^{\beta}, & \text{if } s > 1. \end{cases}$$

Definition 2.2 $(\mathcal{E}, D(\mathcal{E}))$ *satisfies the* cut-off Sobolev inequality CS(Ψ) *on* X*, if there exist constants* $\theta \in (0, 1]$ *and* $c_1, c_2 > 0$ *such that the following holds. For any ball* $B(x_0, 2R) \subset X$*, there exists a cut-off function* $\psi (= \psi_{x_0,R})$ *with the properties:*

(i) $\psi(x) \geq 1$ *for* $x \in B(x_0, R/2)$*.*

(ii) $\psi(x) = 0$ *for* $x \in X \setminus B(x_0, R)$*.*

(iii) $|\psi(x) - \psi(y)| \leq c_1 (d(x, y)/R)^{\theta}$ *for all* $x, y \in X$*.*

(iv) For any ball $B = B(x, s)$ *with* $x \in X$*,* $0 < s \leq R$*, and any* $f \in D(\mathcal{E})$*,*

$$\int_B f^2 d\Gamma(\psi, \psi) \leq c_2 \left(\frac{s}{R}\right)^{2\theta} \left(\int_{2B} d\Gamma(f, f) + \Psi(s)^{-1} \int_{2B} f^2 d\mu \right),$$

where $2B = B(x, 2s)$*.*

Definition 2.3 $(\mathcal{E}, D(\mathcal{E}))$ *satisfies the* Poincaré inequality PI(Ψ) *on* X *if there exists a constant* P_0 *such that for any ball* $B = B(x, R)$ *with* $x \in X$*,* $R > 0$*, and any* $f \in D(\mathcal{E})$*,*

$$\int_B (f - f_B)^2 d\mu \leq P_0 \Psi(R) \int_B d\Gamma(f, f),$$

Here $f_B = \mu(B)^{-1} \int_B f d\mu$*.*

Definition 2.4 $(\mathcal{E}, D(\mathcal{E}))$ *satisfies the* capacity estimate CAP(Ψ) *if there exist constants* $c, C \in (0, \infty)$ *such that for any* $x \in X$*,* $R \geq 0$*,*

$$c \frac{\Psi(R)}{V(x, R)} \leq \left(\mathrm{Cap}_{B(x,2R)}(B(x, R)) \right)^{-1} \leq C \frac{\Psi(R)}{V(x, R)}.$$

For any $x \in X$, $R > 0$, define

$$\begin{aligned} I &= \big(0, 4\Psi(R)\big) \\ B &= B(x, R) \\ Q &= I \times B(x, 2R) \\ Q_- &= \big(\Psi(R), 2\Psi(R)\big) \times B \\ Q_+ &= \big(3\Psi(R), 4\Psi(R)\big) \times B. \end{aligned}$$

Definition 2.5 *The operator $(L, D(L))$ satisfies the* parabolic Harnack inequality PHI(Ψ) *if there exists a constant $H_0 \in (0, \infty)$ such that for any $x \in X$, $R > 0$, any positive function $u \in \mathcal{F}_{\text{loc}}(Q)$ with $(\partial_t - L)u = 0$ weakly in Q has a quasi-continuous modification $\tilde{u}$ so that*

$$\sup_{z \in Q_-} \tilde{u}(z) \leq H_0 \inf_{z \in Q_+} \tilde{u}(z).$$

For $(t, r) \in (0, \infty) \times [0, \infty)$ we consider the two regions

$$\Lambda_1 = \{(t, r) : t \leq 1 \vee r\} \quad \text{ and } \Lambda_2 = \{(t, r) : t \geq 1 \vee r\}.$$

Let

$$h_\beta(r, t) = \exp\left(-\left(\frac{r^\beta}{t}\right)^{1/(\beta-1)}\right).$$

Definition 2.6 $(\mathcal{E}, D(\mathcal{E}))$ *satisfies* HK(Ψ) *if the heat kernel $p(t, x, y)$ on X exists and if there are positive constants c_1, c_2, c_3, c_4 so that*

$$\frac{c_1 h_{\bar\beta}(c_2 d(x, y), t)}{V(x, t^{1/\bar\beta})} \leq p(t, x, y) \leq \frac{c_3 h_{\bar\beta}(c_4 d(x, y), t)}{V(x, t^{1/\bar\beta})},$$

for $x, y \in X$ and $t \in (0, \infty)$ with $(t, d(x, y)) \in \Lambda_1$, and

$$\frac{c_1 h_\beta(c_2 d(x, y), t)}{V(x, t^{1/\beta})} \leq p(t, x, y) \leq \frac{c_3 h_\beta(c_4 d(x, y), t)}{V(x, t^{1/\beta})},$$

for $x, y \in X$ and $t \in (0, \infty)$ with $(t, d(x, y)) \in \Lambda_2$.

Theorem 2.7 *Let $(X, \mu, \mathcal{E}, D(\mathcal{E}))$ be a* MMD *space. The following are equivalent:*

(i) $(\mathcal{E}, D(\mathcal{E}))$ *satisfies* PHI(Ψ).

(ii) $(\mathcal{E}, D(\mathcal{E}))$ *satisfies* HK(Ψ).

(iii) $(\mathcal{E}, D(\mathcal{E}))$ *satisfies* VD, PI(Ψ) *and* CS(Ψ).

Proof. The equivalence of (i) and (ii) is proved in [9][Theorem 5.3] for sufficiently regular solutions, and in [4]. The equivalence of (i) and (iii) is proved in [5][Theorem 2.16]. □

Theorem 2.8 *Let $(X, \mu, \mathcal{E}, D(\mathcal{E}))$ be a* MMD *space. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ satisfies* PHI(Ψ). *Then $(\mathcal{E}, D(\mathcal{E}))$ satisfies* CAP(Ψ).

Proof. See [5]. □

3 Inner uniform domains and Green's function estimates

Let $(X, d, \mu, \mathcal{E}, D(\mathcal{E}))$ be a MMD space that satisfies PHI(Ψ).

3.1 (Inner) uniformity

Let $\Omega \subset X$ be open and connected. The *inner metric* on Ω is defined as

$$d_\Omega(x, y) = \inf \big\{ \mathrm{length}(\gamma) \big| \gamma : [0, 1] \to \Omega \text{ continuous}, \gamma(0) = x, \gamma(1) = y \big\}.$$

Let $\tilde{\Omega}$ be the completion of Ω with respect to d_Ω. Whenever we consider an inner ball $B_{\tilde{\Omega}}(x, R) = \{y \in \tilde{\Omega} : d_\Omega(x, y) < R\}$, we assume that its radius is minimal in the sense that $B_{\tilde{\Omega}}(x, R) \neq B_{\tilde{\Omega}}(x, r)$ for all $r < R$. If x is a point in Ω, denote by $\delta(x) = \delta_\Omega(x) = d(x, \partial\Omega)$ the distance from x to the boundary of Ω.

Definition 3.1 *(i) Let $\gamma : [\alpha, \beta] \to \Omega$ be a rectifiable curve in Ω and let $c \in (0, 1)$, $C \in (1, \infty)$. We call γ a (c, C)-*uniform curve *in Ω if*

$$\delta_\Omega\big(\gamma(t)\big) \geq c \cdot \min\big\{ d\big(\gamma(\alpha), \gamma(t)\big), d\big(\gamma(t), \gamma(\beta)\big) \big\}, \qquad \text{for all } t \in [\alpha, \beta], \quad (2)$$

and if

$$\mathrm{length}(\gamma) \leq C \cdot d\big(\gamma(\alpha), \gamma(\beta)\big).$$

*The domain Ω is called (c, C)-*uniform *if any two points in Ω can be joined by a (c, C)-uniform curve in Ω.*

(ii) Inner uniformity *is defined analogously by replacing the metric d on X with the inner metric d_Ω on Ω.*

(iii) The notion of (inner) (c, C)-length-uniformity *is defined analogously by replacing $d(\gamma(a), \gamma(b))$ by $\mathrm{length}(\gamma\big|_{[a,b]})$.*

The following proposition is taken from [8][Proposition 3.3]. See also [12][Lemma 2.7].

Proposition 3.2 *Assume that (X, d) is a complete, locally compact length metric space with the property that there exists a constant D such that for any $r > 0$, the maximal number of disjoint balls of radius $r/4$ contained in any ball of radius r is bounded above by D. Then any connected open subset $U \subset X$ is uniform if and only if it is length-uniform.*

Lemma 3.3 *Let Ω be a (c_u, C_u)-inner uniform domain in (X, d). For every ball $B = B_{\tilde{\Omega}}(x, r)$ in $(\tilde{\Omega}, d_\Omega)$ with minimal radius, there exists a point $x_r \in B$ with $d_\Omega(x, x_r) = r/4$ and $d(x_r, \tilde{\Omega} \setminus \Omega) \geq c_u r/8$.*

Proof. See [8][Lemma 3.20]. □

The following lemma is crucial for the proof of the boundary Harnack principle on inner uniform domains rather than uniform domains. A version of this lemma was already used in [2] to prove a boundary Harnack principle on inner uniform domains in Euclidean space.

For any ball $D = B(x,r)$ with $x \in \tilde{\Omega}$, let D' be the connected component of $D \cap \tilde{\Omega}$ that contains x.

Lemma 3.4 *Let Ω be a (c_u, C_u)-inner uniform domain in (X,d,μ) and assume that μ has the volume doubling property with constant D_0. Then there exists a positive constant C_0 such that for any ball $D = B(x,r)$ with $x \in \tilde{\Omega}$, $r > 0$,*

$$B_{\tilde{\Omega}}(x,r) \subset D' \subset B_{\tilde{\Omega}}(x, C_0 r).$$

The constant C_0 depends only on D_0, c_u, C_u.

Proof. See [10][Lemma 3.7]. □

3.2 Green's function estimates

Theorem 3.5 *(i) For any fixed $\epsilon \in (0,1)$ there are constants $c_1, C_1 \in (0,\infty)$ such that for any ball $B = B(a,R) \subset X$ and any $x, y \in B(a,(1-\epsilon)R)$ and $0 < \epsilon t \le \Psi(R)$, the Dirichlet heat kernel p_B^D, is bounded below by*

$$p_B^D(t,x,y) \ge \frac{C_1}{V(x, t^{1/\bar{\beta}} \wedge R_x)} h_{\bar{\beta}}(c_1 d(x,y), t) \quad t \le 1 \vee d(x,y), \tag{3}$$

$$p_B^D(t,x,y) \ge \frac{C_1}{V(x, t^{1/\beta} \wedge R_x)} h_{\beta}(c_1 d(x,y), t) \quad t \ge 1 \vee d(x,y), \tag{4}$$

where $R_x = d(x, \partial B)$.

(ii) For any fixed $\epsilon \in (0,1)$ there are constants $c_2, C_2 \in (0,\infty)$ such that for any ball $B = B(a,R) \subset X$ and any $x,y \in B(a,R)$, $t \ge \epsilon\Psi(R)$, the Dirichlet heat kernel p_B^D is bounded above by

$$p_B^D(t,x,y) \le \frac{C_2}{V(x,R)} \exp(-c_2 t/\Psi(R)). \tag{5}$$

(iii) There exist constants $c_3, C_3 \in (0,\infty)$ such that for any ball $B = B(a,R) \subset X$ and any $x,y \in B(a,R)$, the Dirichlet heat kernel p_B^D is bounded above by

$$p_B^D(t,x,y) \le \frac{C_3 h_{\bar{\beta}}(c_3 d(x,y), t)}{V(x, t^{1/\bar{\beta}})}, \quad t \le 1 \vee d(x,y) \tag{6}$$

$$p_B^D(t,x,y) \le \frac{C_3 h_{\beta}(c_3 d(x,y), t)}{V(x, t^{1/\beta})}, \quad t \ge 1 \vee d(x,y). \tag{7}$$

All the constants c_i, C_i above depend only on H_0, β, $\bar{\beta}$.

Proof of Theorem 3.5. Fix a ball $B = B(a, R) \subset X$. (iii) Let $x, y \in B(a, R)$. By Theorem 2.7, we have for the heat kernel $p(t, x, y)$ on X associated with $(\mathcal{E}, D(\mathcal{E}))$,

$$p(t,x,y) \le \frac{c_3 h_{\bar\beta}(c_4 d(x,y), t)}{V(x, t^{1/\bar\beta})}, \quad t \le 1 \vee d(x,y) \tag{8}$$

$$p(t,x,y) \le \frac{c_3 h_{\beta}(c_4 d(x,y), t)}{V(x, t^{1/\beta})}, \quad t \ge 1 \vee d(x,y). \tag{9}$$

Thus, assertion (iii) follows from the set monotonicity of the kernel p_B^D.

To show the on-diagonal estimate in (i) we follow [13][Proof of Theorem 5.4.10]. Let $x \in B(a, (1-\epsilon)R)$ and $0 < \epsilon t \le \Psi(R)$. Let $t' = \frac{t}{4} \wedge \Psi(R_x/8)$. Then $B_x = B(x, 8\Psi^{-1}(t'))$ lies in $B(a, R)$. By the cut-off Sobolev inequality which holds by Theorem 2.7, there exists a cut-off function ψ such that $0 \le \psi \le 1$, $\psi = 1$ on $B(x, 2\Psi^{-1}(t'))$ and $\psi = 0$ on $X \setminus B(x, 4\Psi^{-1}(t'))$. Let

$$u(t,y) = \begin{cases} P^D_{B_x,t}\psi(y) & \text{if } t > 0, \\ \psi(y) & \text{if } t \le 0. \end{cases}$$

Then u is a local weak solution of

$$Lu = \frac{\partial}{\partial t} u \quad \text{on } Q' = (-\infty, +\infty) \times B(x, 2\Psi^{-1}(t')).$$

Applying the parabolic Harnack inequality to u and then to $p^D_{B_x}(\cdot, x, \cdot)$, we get

$$\begin{aligned} 1 = u(0,x) &\le C\, u(t/2, x) \\ &= C \int p^D_{B_x}(t/2, x, y)\psi(y)\mu(dy) \\ &\le C \int_{B(x, 4\Psi^{-1}(t'))} p^D_{B_x}(t/2, x, y)\mu(dy) \\ &\le C'\, V(x, 4\Psi^{-1}(t'))\, p^D_{B_x}(t, x, x). \end{aligned}$$

Using the set monotonicity of the kernel and the doubling property of the function Ψ and the volume, we get

$$p^D_B(t,x,x) \ge p^D_{B_x}(t,x,x) \ge C'' \frac{1}{V(x, \Psi^{-1}(t) \wedge R_x)}.$$

The off-diagonal estimate now follows from [9][Lemma 5.1].

(ii) follows from changing notation in [9][Lemma 5.13 part 3]. □

Lemma 3.6 *For any relatively compact open set $V \subset X$, the Green function $y \mapsto G_V(x, y)$ is in $\mathcal{F}^0_{\mathrm{loc}}(V, V \setminus \{x\})$ for any fixed $x \in V$.*

Proof. Let $\Omega \subset V$ be open and relatively compact in $\bar{V}$ with $d(\Omega, x) > 0$. Pick finitely many balls $B_i = B(z_i, s_i)$ so that $\Omega \subset \bigcup_i B_i$ and $K = \bigcup_i \overline{B(z_i, 4s_i)} \subset X \setminus \{x\}$. For each i, there exists by the cut-off Sobolev inequality a continuous cut-off function ψ_i with $\psi = 1$ on B_i and $\psi_i = 0$ on $X \setminus B(z_i, 2s_i)$. Let $\psi(y) = \min\{1, \sum_i \psi_i(y)\}$ for all $y \in K$, and $f^\sharp = \psi G_V(x, \cdot)$. Then $f^\sharp = G_V(x, \cdot)$ on Ω. It remains to show that $f^\sharp$ is in $\mathcal{F}^0(V)$.

Recall that the map $y \mapsto p_V^D(t, x, \cdot)$ is in $\mathcal{F}^0(V)$. The heat kernel upper bounds of Theorem 3.5 imply that $\psi G_V(x, \cdot) \in L^2(X, \mu)$. Indeed, $p_V^D(t, x, y) \leq p_B^D(t, x, y)$ with $B = B(x, R)$ and R is chosen large enough so that $V \subset B$. By Theorem 3.5, there are constants $c, C(B) \in (0, \infty)$ such that for all $t \geq \Psi(R)$ and $x, y \in V$,

$$p_V^D(t, x, y) \leq C(B) e^{-ct/\Psi(R)}, \tag{10}$$

and there are constants $c', C' \in (0, \infty)$ depending on V, K, H_0, β, $\bar{\beta}$ such that for all $t > 0$ and $x, y \in V \cap K$,

$$p_V^D(t, x, y) \leq C' e^{-c'/t^*}, \tag{11}$$

where $t^* = t^{1/(\bar{\beta}-1)}$ if $t \leq 1 \vee d(x, y)$ and $t^* = t^{1/(\beta-1)}$ if $t > 1 \vee d(x, y)$. This shows that the integral $\psi G_V(x, \cdot) = \int_0^\infty \psi p_V^D(t, x, \cdot) dt$ converges at 0 and ∞ in $L^2(X, \mu)$. Hence $\psi G_V(x, \cdot)$ is in $L^2(X, \mu)$.

For fixed $0 < a < b < \infty$, set $g = \int_a^b p_V^D(t, x, \cdot) dt$ and observe that $\psi g, \psi^2 g \in \mathcal{F}^0(V)$. Thus, using the cut-off Sobolev inequality, we obtain

$$\begin{aligned}
\int_V d\Gamma(\psi g, \psi g) &\leq 2 \int_{K \cap V} \psi^2 d\Gamma(g, g) + 2 \int_{K \cap V} g^2 d\Gamma(\psi, \psi) \\
&\leq 2 \int_{K \cap V} d\Gamma(g, g) + C \sum_i \int_{B(z_i, 2s_i) \cap V} g^2 d\Gamma(\psi_i, \psi_i) \\
&\leq C' \int_{K \cap V} g(-Lg)\, d\mu + C \sup_i \Psi(2s_i)^{-1} \int_{B(z_i, 4s_i) \cap V} g^2 d\mu \\
&\leq C' \int_{K \cap V} g\big(p_V^D(a, x, \cdot) - p_V^D(b, x, \cdot)\big) d\mu + C \int_{K \cap V} g^2 d\mu \\
&\leq C' \int_{K \cap V} g p_V^D(a, x, \cdot) d\mu + C \int_{K \cap V} g^2 d\mu.
\end{aligned}$$

The constants C and C' change from line to line and depend on Ω and Ψ. Now, observe that (10)-(11) imply that

$$\int_{K \cap V} g^2 d\mu = \int_{K \cap V} \left(\int_a^b p_V^D(t, x, \cdot) dt \right)^2 d\mu$$

tends to 0 when a, b tend to infinity or when a, b tend to 0 (this is indeed the argument we used above to show that $G_V(x, \cdot)$ is in $L^2(X, d\mu)$). The same estimates (10)-(11) imply that $\int_{K \cap V} g p_V^D(a, x, \cdot) d\mu$ tends to 0 when a, b tend to infinity or when a, b tend to 0. This implies that the integral $\psi G_V(x, y) = \psi \int_0^\infty p_V^D(t, x, \cdot) dt$ converges in $\mathcal{F}^0(V)$ as desired. □

Lemma 3.7 *(i) There is a constant C_1 depending only on H_0, β, $\bar\beta$, such that for any ball $B(z,R) \subset X$, and any $x, y \in B(z,R)$ with $d(x,y) \geq (\epsilon R)$ for some $\epsilon > 0$,*

$$G_{B(z,R)}(x,y) \leq C_1 \frac{\Psi(R)}{V(x,R)}. \tag{12}$$

(ii) Fix $\theta \in (0,1)$. There is a constant C_2 depending only on H_0, β, $\bar\beta$, θ, such that for any ball $B(z,R) \subset X$,

$$\forall x, y \in B(z,\theta R), \quad G_{B(z,R)}(x,y) \geq C_2 \frac{\Psi(R)}{V(x,R)}. \tag{13}$$

Proof. The lower bound (13) follows from integrating (4) from $t = \Psi(\theta R)$ to $t = \Psi(R)$ (if $R \leq 1$) and from $t = \Psi(2R)$ to $t = \Psi(3R)$ (if $R > 1$). For the upper bound (12) we use the heat kernel estimates of Theorem 3.5,

$$p_B^D(t,x,y) \leq \begin{cases} \frac{C_3 h_{\bar\beta}(c_3 d(x,y),t)}{V(x,t^{1/\bar\beta})}, & t \leq 1 \vee d(x,y),\ t \leq \Psi(R), \\ \frac{C_3 h_{\beta}(c_3 d(x,y),t)}{V(x,t^{1/\beta})}, & t \geq 1 \vee d(x,y),\ t \leq \Psi(R), \\ \frac{C_2}{V(x,R)} \exp(-c_2 t/\Psi(R)), & t > \Psi(R), \end{cases}$$

where $B = B(z,R)$. Then

$$\begin{aligned} G_{B(z,R)}(x,y) &\leq \int_0^{\Psi(R)} p_B^D(t,x,y)dt + \int_{\Psi(R)}^{\infty} \frac{C_2}{V(x,R)} \exp(-c_2 t/\Psi(R))dt \\ &\leq \int_0^{\Psi(R)} p_B^D(t,x,y)dt + C\frac{\Psi(R)}{V(x,R)}, \end{aligned}$$

for some constant $C > 0$. It remains to estimate the integral on the right hand side. In the case $R \leq 1$ it is $\Psi(R) = R^{\bar\beta}$. Due to the volume doubling condition, we have for some $\nu = \nu(D_0) > 2\bar\beta$,

$$\begin{aligned} \int_0^{R^{\bar\beta}} p_B^D(t,x,y)dt &\leq \frac{C}{V(x,R)} \int_0^{R^{\bar\beta}} \left(\frac{R^{\bar\beta}}{t}\right)^{\frac{\nu}{\bar\beta}} \exp\left(-c\left(\frac{(\epsilon R)^{\bar\beta}}{t}\right)^{1/(\bar\beta-1)}\right)dt \\ &\leq \frac{CR^{\bar\beta}}{V(x,R)}, \end{aligned}$$

since the integrand in the second integral is bounded by a constant independent of R. Here the constants $c, C > 0$ change from line to line and depend only on H_0, β, $\bar\beta$.

Now consider the case $R > 1$ and $d(x,y) \leq \Psi(R)$. Then

$$\begin{aligned} &\int_0^{1\vee d(x,y)} p_B^D(t,x,y)dt \\ \leq &\frac{C}{V(x,R)} \int_0^{1\vee d(x,y)} \left(\frac{R^{\bar\beta}}{t}\right)^{\nu} \exp\left(-c\left(\frac{(\epsilon R)^{\bar\beta}}{t}\right)^{1/(\bar\beta-1)}\right)dt \end{aligned}$$

$$\leq C\frac{1 \vee d(x,y)}{V(x,R)} \leq \frac{C\,R^{\beta}}{V(x,R)}.$$

Here we used that the integrand is bounded from above. This can be seen by considering the cases $d(x,y) \leq R^{\bar\beta}$ and $d(x,y) > R^{\bar\beta}$ separately. Moreover,

$$\int_{1\vee d(x,y)}^{R^{\beta}} p_B^D(t,x,y)dt \leq \frac{C}{V(x,R)} \int_{1\vee d(x,y)}^{R^{\beta}} \left(\frac{R^{\beta}}{t}\right)^{\nu} \exp\left(-c\left(\frac{(\epsilon R)^{\beta}}{t}\right)^{1/(\beta-1)}\right)dt$$
$$\leq \frac{C\,R^{\beta}}{V(x,R)}.$$

The case $R > 1$ and $d(x,y) \geq \Psi(R)$ can be treated similarly. □

Lemma 3.8 *Fix $\theta \in (0,1)$. Let $U \subset X$ be open.*

(i) Let $z \in X$, $R > 0$. There is a constant C_1 depending only on θ, H_0, β, $\bar\beta$, such that

$$G_{B_U(z,R)}(x,y) \leq G_{U\cap B(z,R)}(x,y) \leq C_1 \frac{\Psi(R)}{V(x,R)}, \tag{14}$$

for all $x, y \in U \cap B(z,R)$ with $d(x,y) \geq (\theta R)$.

(ii) Let $z \in U$, $R > 0$, $\delta \in (0,1/3)$. Suppose that $U \cap B(z,2R)$ has minimal radius and that any two points in $B_U(z,\delta R)$ can be connected by a (c_u, C_u)-inner uniform curve in U. Then there is a constant C_2 depending only on θ, H_0, β, $\bar\beta$, c_u, C_u such that

$$G_{B_U(z,R)}(x,y) \geq C_2 \frac{\Psi(R)}{V(x,R)}, \tag{15}$$

for all $x, y \in B_U(z,\delta R)$ with $d(x, X\setminus U)$, $d(y, X \setminus U) \in (\theta R, \infty)$ and $d_U(x,y) \leq \delta R/C_u$.

Proof. We follow the line of reasoning of [8][Lemma 4.9]. (i) Set $B = B(z,R)$, $W = U \cap B(z,R)$. The upper bound (14) follows easily from Lemma 3.7 and the monotonicity inequality $G_W \leq G_B$. (ii) By assumption, there is an $\epsilon_1 > 0$ such that, for any $x, y \in B_U(z,\delta R)$ satisfying $d(x, X \setminus U)$, $d(y, X \setminus U) > \theta R$, there is a path in U from x to y of length less than $C_u d_U(x,y) \leq \delta R$ that stays at distance at least $\epsilon_1 R$ from $X \setminus U$. Since $x, y \in B_U(z,\delta R)$ and $\delta < 1/3$, this path is contained in

$$B_U(z,R) \cap \{\zeta \in U : d(\zeta, X \setminus U) > \epsilon_1 R\}.$$

Using this path, the Harnack inequality easily reduces the lower bound (15) to the case when y satisfies $d(x,y) = \eta R$ for some arbitrary fixed $\eta \in (0,\epsilon_1)$ small enough. Pick $\eta > 0$ so that, under the conditions of the Lemma, the ball $B_x = B(x, 2\eta R)$ is contained in $B_U(z,R)$. Let $W = B_U(z,R)$. Then the

monotonicity property of Green functions implies that $G_W(x,y) \geq G_{B_x}(x,y)$. Lemma 3.7 and the volume doubling property then yield

$$G_W(x,y) \geq C_2 \frac{\Psi(R)}{V(x,R)}.$$

This is the desired lower bound. □

4 Boundary Harnack Principle

4.1 Reduction to Green functions estimates

Let $(X, d, \mu, \mathcal{E}, D(\mathcal{E}))$ be a MMD space that satisfies PHI(Ψ). We obtain that under these assumptions, local weak solutions in Y of $Lu = 0$ are harmonic functions for the associated Markov process and satisfy the maximum principle. This can be proved following the line of reasoning given in [7][Theorem 4.3.2, Lemma 4.3.2] and using [11][Proposition V.1.6, Proof of Lemma III.1.4].

Let $\Omega \subset X$ be open and connected. For $\xi \in \partial\Omega$, we denote the inner ball $B_{\tilde{\Omega}}(\xi, r) \cap \Omega$ by $B_\Omega(\xi, r)$. Let $c_u \in (0,1)$ and $C_u \in (1,\infty)$. Let $A_3 = 2(12+12C_u)$, $A_0 = A_3 + 7$. For $\xi \in \partial\Omega$, let R_ξ be the largest radius so that

(i) $12R_\xi/c_u \leq \mathrm{diam}_\Omega(\Omega)/2$ if Ω is a bounded domain.

(ii) Any two points in $B_{\tilde{\Omega}}(\xi, 12R_\xi/c_u)$ can be connected by a curve that is (c_u, C_u)-inner uniform in Ω.

Theorem 4.1 *There exists a constant $A_1' \in (1,\infty)$ such that for any $\xi \in \partial\Omega$ with $R_\xi > 0$ and any*

$$0 < r < \inf\{R_{\xi'} : \xi' \in \partial\Omega \cap B_{\tilde{\Omega}}(\xi, 7R_\xi)\},$$

we have

$$\frac{G_{Y'}(x,y)}{G_{Y'}(x',y)} \leq A_1' \frac{G_{Y'}(x,y')}{G_{Y'}(x',y')},$$

for all $x, x' \in \Omega \cap B_{\tilde{\Omega}}(\xi, r)$ and $y, y' \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$. Here $Y' = B_{\tilde{\Omega}}(\xi, A_0 r)$. The constant A_1' depends only on H_0, β, $\bar{\beta}$, c_u, C_u.

The proof of this theorem is the content of Section 4.2 below. It is based on the estimates for the Green's functions in Section 3.2.

Theorem 4.2 *There exists a constant $A_1 \in (1,\infty)$ such that for any $\xi \in \partial\Omega$ with $R_\xi > 0$, any*

$$0 < r < \inf\{R_{\xi'} : \xi' \in \partial\Omega \cap B_{\tilde{\Omega}}(\xi, 7R_\xi)\},$$

and any two non-negative weak solutions u, v of $Lu = 0$ in $Y' = B_{\tilde{\Omega}}(\xi, A_0 r)$ with weak Dirichlet boundary condition along $(\partial\Omega) \cap B_{\tilde{\Omega}}(\xi, 6r)$, we have

$$\frac{u(x)}{u(x')} \leq A_1 \frac{v(x)}{v(x')},$$

for all $x, x' \in B_\Omega(\xi, r)$. The constant A_1 depends only on H_0, β, $\bar{\beta}$, c_u, C_u.

Proof. Fix $\xi \in \partial\Omega$ and $r > 0$ as in the theorem. Following the argument given in [1][Proof of Theorem 1], we show that for any weak solution u of $Lu = 0$ in Y' with weak Dirichlet boundary condition along $(\partial\Omega) \cap B_{\tilde{\Omega}}(\xi, 6r)$, there exists a Borel measure ν_u such that

$$u(x) = \int_{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)} G_{Y'}(x, y) d\nu_u(y) \tag{16}$$

for all $x \in \Omega \cap B_{\tilde{\Omega}}(\xi, r)$, where $G_{Y'}$ is the Green's function corresponding to the Dirichlet form $(\mathcal{E}^D_{Y'}, D(\mathcal{E}^D_{Y'}))$. Let $\hat{R}_u^{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)}$ be the lower regularization of the reduced function

$$\begin{aligned} R_u^{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)}(x) = \inf\{ v(x) :\, & v \text{ positive and } L\text{-superharmonic on } Y', \\ & v \geq u \text{ on } \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)\} \end{aligned}$$

of u on $\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$. Since u is a positive local weak solution of $Lu = 0$ on Y', $u = \hat{R}_u^{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)}$ quasi-everywhere on $\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$, and $L\hat{R}_u^{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)} = 0$ on $\Omega \setminus \partial B_{\tilde{\Omega}}(\xi, 6r)$. Moreover, $\hat{R}_u^{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)} = 0$ q.e. on $(\partial\Omega) \cap B_{\tilde{\Omega}}(\xi, 6r)$ by assumption. Hence $u = \hat{R}_u^{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)}$ on $\Omega \cap B_{\tilde{\Omega}}(\xi, 6r)$ by the maximum principle. As in [3][Proof of Theorem 5.3.5], one can show that there is a measure ν_u supported on $\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$, so that

$$\hat{R}_u^{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)}(x) = \int_{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)} G_{Y'}(x, y) d\nu_u(y), \quad \forall x \in \Omega \cap B_{\tilde{\Omega}}(\xi, r).$$

This proves (16).

By Theorem 4.1 below, there exists a constant $A_1' \in (1, \infty)$ such that for all $x, x' \in B_\Omega(\xi, r)$ and all $y, y' \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$, we have

$$\frac{G_{Y'}(x, y)}{G_{Y'}(x', y)} \leq A_1' \frac{G_{Y'}(x, y')}{G_{Y'}(x', y')}.$$

For any (fixed) $y' \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$, we find that

$$\begin{aligned} \frac{1}{A_1'} u(x) &\leq \frac{G_{Y'}(x, y')}{G_{Y'}(x', y')} \int_{\Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)} G_{Y'}(x', y) d\nu_u(y) \\ &= \frac{G_{Y'}(x, y')}{G_{Y'}(x', y')} u(x') \leq A_1' u(x). \end{aligned}$$

We get a similar inequality for v. Thus, for all $x, x' \in B_\Omega(\xi, r)$,

$$\frac{1}{A_1'} \frac{u(x)}{u(x')} \leq \frac{G_{Y'}(x, y')}{G_{Y'}(x', y')} \leq A_1' \frac{v(x)}{v(x')}. \tag{17}$$

□

4.2 Proof of Theorem 4.1

We follow closely [1], [8] and [10]. Let Ω as above and fix $\xi \in \partial\Omega$ with $R_\xi > 0$.

Definition 4.3 *For $\eta \in (0,1)$ and any open $U \subset X$, define the* capacity width *$w_\eta(U)$ by*

$$w_\eta(U) = \inf\left\{ r > 0 : \forall x \in U, \frac{\mathrm{Cap}_{B(x,2r)}\left(\overline{B(x,r)} \setminus U\right)}{\mathrm{Cap}_{B(x,2r)}\left(\overline{B(x,r)}\right)} \geq \eta \right\}.$$

Note that $w_\eta(U)$ is a decreasing function of $\eta \in (0,1)$ and an increasing function of the set U.

Lemma 4.4 *There are constants $A_7, \eta \in (0,\infty)$ depending only on H_0, β, $\bar\beta$, c_u, C_u such that for all $0 < r < 2R_\xi$,*

$$w_\eta(\{y \in Y' : d_\Omega(y,\partial\Omega) < r\}) \leq A_7 r.$$

Proof. We follow [8][Lemma 4.12]. Let $Y_r = \{y \in Y' : d(y,\partial\Omega) < r\}$ and $y \in Y_r$. Since $r < c_u \mathrm{diam}_\Omega(\Omega)/12$, there exists a point $x \in \Omega$ such that $d_\Omega(x,y) \geq 4r/c_u$. There exists an inner uniform curve connecting y to x in Ω. Let $z \in \Omega \cap \partial B_\Omega(y, 2r/c_u)$ be a point on this curve and note that $d_\Omega(y,z) = 2r/c_u \leq d_\Omega(x,y) - d_\Omega(y,z) \leq d_\Omega(x,z)$. Hence,

$$d_\Omega(z,\partial\Omega) \geq c_u \min\{d_\Omega(y,z), d_\Omega(z,x)\} \geq 2r.$$

So for any $y \in Y_r$ there exists a point $z \in \Omega \cap \partial B_\Omega(y, 2r/c_u)$ with $d(z,\partial\Omega) \geq 2r$. Thus, $B(z,r) \subset B(y, A_7 r) \setminus Y_r$ if $A_7 = 2/c_u + 1$. The capacity of $B(y, A_7 r) \setminus Y_r$ in $B(y, 2A_7 r)$ is larger than the capacity of $B(z,r)$ in $B(y, 2A_7 r)$, which is larger than the capacity of $B(z,r)$ in $B(z, 3A_7 r)$. This is comparable to $V(z,r)/\Psi(r)$, by an argument similar to the proof of CAP(Ψ). Hence, $w_\eta(Y_r) \leq \mathrm{const} \cdot A_7 r$ for some $\eta > 0$. $\square$

Fix $\eta \in (0,1)$ small enough so that the conclusion of Lemma 4.4 applies and write $w(U) := w_\eta(U)$ for the capacity width of an open set $U \subset \Omega$.

The following lemma relates the capacity width to the L-harmonic measure ω. We write $f \asymp g$ to indicate that $C_1 g \leq f \leq C_2 g$, for some constants C_1, C_2 that only depend on H_0, β, $\bar\beta$, c_u, C_u.

Lemma 4.5 *There is a constant $a_1(H_0, \beta, \bar\beta)$ such that for any non-empty open set $U \subset X$ and any $x \in U$, $r > 0$, we have*

$$\omega_{U \cap B(x,r)}(x, U \cap \partial B(x,r)) \leq \exp(2 - a_1 r / w(U)).$$

Proof. We follow [1][Lemma 1] and [8][Lemma 4.13]. We may assume that $r/w(U) > 2$. For any $\kappa \in (0,1)$, we can pick $w(U) \leq s < w(U) + \kappa$ so that

$$\frac{\mathrm{Cap}_{B(y,2s)}(\overline{B(y,s)} \setminus U)}{\mathrm{Cap}_{B(y,2s)}(\overline{B(y,s)})} \geq \eta \quad \forall y \in U.$$

Fix $y \in U$ and let $E = \overline{B(y,s)} \setminus U$. Let ν_E be the equilibrium measure of E in $B = B(y, 2s)$. We claim that there exists $A_2 > 0$ such that

$$G_B \nu_E \geq A_2 \eta \quad \text{on } B(y,s). \tag{18}$$

Let $F = \overline{B(y,s)}$ and ν_F be the equilibrium measure of F in B. Then, by the Harnack inequality, for any z with $d(y,z) = 3s/2$, we have

$$G_B(z,\zeta) \asymp G_B(z,y) \quad \forall \zeta \in B(y,s).$$

Hence,

$$G_B \nu_F(z) = \int_F G_B(z,\zeta)\nu_F(d\zeta) \asymp G_B(z,y)\nu_F(F)$$

and

$$G_B \nu_E(z) = \int_E G_B(z,\zeta)\nu_E(d\zeta) \asymp G_B(z,y)\nu_E(E).$$

Moreover, since $\nu_F(F) = \mathrm{Cap}_B(F)$, estimate CAP($\Psi$) and Lemma 3.7 yield that $G_B \nu_F(z) \simeq 1$. Hence, by definition of s, for any $z \in \partial B(y, 3s/2)$,

$$G_B \nu_E(z) \asymp \frac{G_B \nu_E(z)}{G_B \nu_F(z)} \asymp \frac{\nu_E(E)}{\nu_F(F)} \asymp \frac{\mathrm{Cap}_B(E)}{\mathrm{Cap}_B(F)} \geq \eta.$$

This proves (18).

Let $x \in U$. For simplicity, write $\omega(\cdot) = \omega_{U \cap B(x,r)}(\cdot, U \cap \partial B(x,r))$. Let k be the integer such that $2kw(U) < r < 2(k+1)w(U)$, and pick $s > w(U)$ so close to $w(U)$ that $2ks < r$. We claim that

$$\sup_{U \cap B(x, r-2js)} \{\omega\} \leq (1 - A_2\eta)^j \tag{19}$$

for $j = 0, 1, \ldots, k$ with A_2, η as in (18). Note that for $j = k$, (19) yields the inequality stated in this Lemma:

$$\omega(x) \leq (1 - A_2\eta)^k \leq \exp\left(\log\left((1 - A_2\eta)^{\frac{r}{2w(U)}}\right)\right) \leq e^2 \exp(-a_1 r / w(U)),$$

with $a_1 = -(\log(1 - A_2\eta))/2$.

Inequality (19) is proved by induction, starting with the trivial case $j = 0$. Assume that (19) holds for $j-1$. By the maximum principle, it suffices to prove

$$\sup_{U \cap \partial B(x, r-2js)} \{\omega\} \leq (1 - A_2\eta)^j. \tag{20}$$

Let $y \in U \cap \partial B(x, r - 2js)$. Then $B(y, 2s) \subset B(x, r - 2(j-1)s)$ so that the induction hypothesis implies that

$$\omega \leq (1 - A_2\eta)^{j-1} \quad \text{on } U \cap B(y, 2s).$$

Since ω vanishes (quasi-everywhere) on $\partial U \cap B(x,r) \supset \partial U \cap B(y, 2s)$, the maximum principle implies that

$$\omega(b) = \int_{\partial(U \cap B(y,2s))} \omega(a) \omega_{U \cap B(y,2s)}(b, da)$$

$$\leq (1 - A_2\eta)^{j-1}\omega_{U\cap B(y,2s)}(b, U \cap \partial B(y, 2s))$$

for any $b \in V \cap B(y, 2s)$. To estimate

$$u = \omega_{U\cap B(y,2s)}(\cdot, U \cap \partial B(y, 2s)),$$

on $U \cap B(y, 2s)$, we compare it to

$$v = 1 - G_{B(y,2s)}\nu_E,$$

where, as above, ν_E denotes the equilibrium measure of $E = \overline{B(y, 2s)} \setminus U$ in $B(y, 2s)$. Both functions are L-harmonic in $U \cap B(y, 2s)$ and $u \leq v$ on $\partial(U \cap B(y, 2s))$ quasi-everywhere (in the limit sense). By (18), this implies

$$u = \omega_{U\cap B(y,2s)}(\cdot, U \cap \partial B(y, 2s)) \leq v \leq 1 - A_2\eta$$

on $U \cap B(y, 2s)$. Hence,

$$\omega \leq (1 - A_2\eta)^j \quad \text{on } U \cap B(y, 2s).$$

Since this holds for any $y \in U \cap \partial B(x, r - 2js)$, (20) is proved. $\square$

Lemma 4.6 *There exist constants $A_2, A_3 \in (0, \infty)$ depending only on H_0, β, $\bar{\beta}$, c_u, C_u, such that for any $0 < r < R_\xi$ and any $x \in B_{\tilde{\Omega}}(\xi, r)$, we have*

$$\omega_{B_{\tilde{\Omega}}(\xi,2r)}\big(x, \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 2r)\big) \leq A_2 \frac{V(\xi, r)}{\Psi(r)} G_{B_{\tilde{\Omega}}(\xi,C_0A_3r)}(x, \xi_{16r}).$$

Here ξ_{16r} is any point in Ω with $d_\Omega(\xi, \xi_{16r}) = 4r$ and

$$\delta(\xi_{16r}) := d(\xi_{16r}, X \setminus \Omega) = d(\xi_{16r}, X \setminus Y') \geq 2c_u r.$$

Proof. We follow [10][Lemma 4.7]. Let $A_3 = 2(12 + 12C_u)$ so that all (c_u, C_u)-inner uniform paths connecting two points in $B_{\tilde{\Omega}}(\xi, 12r)$ stay in $B_{\tilde{\Omega}}(\xi, A_3r/2)$. Recall that $Y' = B_{\tilde{\Omega}}(\xi, A_0r)$, where $A_0 = A_3 + 7$. For any $z \in B_{\tilde{\Omega}}(\xi, A_3r)$, set

$$G'(z) = G_{B_{\tilde{\Omega}}(\xi,A_3r)}(z, \xi_{16r}).$$

Let $s = \min\{c_ur, 5r/C_u\}$. As

$$B_{\tilde{\Omega}}(\xi_{16r}, s) \subset B_{\tilde{\Omega}}(\xi, A_3r) \setminus B_{\tilde{\Omega}}(\xi, 2r),$$

the maximum principle yields

$$\forall y \in B_{\tilde{\Omega}}(\xi, 2r), \quad G'(y) \leq \sup_{z\in\partial B_{\tilde{\Omega}}(\xi_{16r},s)} G'(z).$$

Lemma 3.8 and the volume doubling condition yield

$$\sup_{z\in\partial B_{\tilde{\Omega}}(\xi_{16r},s)} G'(z) \leq C\frac{\Psi(r)}{V(\xi, r)},$$

for some constant $C > 0$. Hence, there exists $\epsilon_1 > 0$ such that

$$\forall y \in B_{\tilde{\Omega}}(\xi, 2r), \quad \epsilon_1 \frac{V(\xi, r)}{\Psi(r)} G'(y) \leq \mathrm{e}^{-1}.$$

Write

$$B_{\tilde{\Omega}}(\xi, 2r) = \bigcup_{j \geq 0} U_j \cap B_{\tilde{\Omega}}(\xi, 2r), \tag{21}$$

where

$$U_j = \left\{ x \in Y' : \exp(-2^{j+1}) \leq \epsilon_1 \frac{V(\xi, r)}{\Psi(r)} G'(x) < \exp(-2^j) \right\}.$$

Let $V_j = \left(\bigcup_{k \geq j} U_k \right) \cap B_{\tilde{\Omega}}(\xi, 2r)$. We claim that

$$w_\eta(V_j) \leq A_4 r \exp\left(-2^j/\sigma\right) \tag{22}$$

for some constants $A_4, \sigma \in (0, \infty)$.

Suppose $x \in V_j$. Observe that for $z \in \partial B_{\tilde{\Omega}}(\xi_{16r}, s)$, by the inner uniformity of the domain, the length of the Harnack chain of balls in $B_{\tilde{\Omega}}(\xi, A_3 r) \setminus \{\xi_{16r}\}$ connecting x to z is at most $A_5 \log(1 + A_6 r / d(x, X \setminus Y'))$ for some constants $A_5, A_6 \in (0, \infty)$. Therefore there are constants $\epsilon_2, \epsilon_3, \sigma > 0$ such that

$$\begin{aligned} \exp(-2^j) > \epsilon_1 \frac{V(\xi, r)}{\Psi(r)} G'(x) &\geq \epsilon_2 \frac{V(\xi, r)}{\Psi(r)} G'(z) \left(\frac{d(x, X \setminus Y')}{r} \right)^\sigma \\ &\geq \epsilon_3 \left(\frac{d(x, X \setminus Y')}{r} \right)^\sigma. \end{aligned}$$

The last inequality is obtained by applying 3.8 with $R = A_3 r$ and $\delta = 5/A_3$. Now we have that for any $x \in V_j$,

$$d(x, X \setminus \Omega) = d(x, X \setminus Y') \leq \left(\epsilon_3^{-1/\sigma} \exp(-2^j/\sigma) r \right) \wedge 2r.$$

This together with Lemma 4.4 yields (22).

Let $R_0 = 2r$ and for $j \geq 1$,

$$R_j = \left(2 - \frac{6}{\pi^2} \sum_{k=1}^{j} \frac{1}{k^2} \right) r.$$

Then $R_j \downarrow r$ and

$$\begin{aligned} \sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{a_1 (R_{j-1} - R_j)}{A_4 r \exp(-2^j/\sigma)} \right) &= \sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{6 a_1}{A_4 \pi^2} j^{-2} \exp(2^j/\sigma) \right) \\ &\leq \sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{3 a_1}{C_0 A_4 \pi^2} j^{-2} \exp(2^j/\sigma) \right) \end{aligned}$$

$$< C < \infty. \tag{23}$$

Let $\omega_0 = \omega(\cdot, \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 2r), B_{\tilde{\Omega}}(\xi, 2r))$ and

$$d_j = \begin{cases} \sup\left\{\frac{\Psi(r)\omega_0(x)}{V(\xi,r)G'(x)} : x \in U_j \cap B_{\tilde{\Omega}}(\xi, R_j)\right\}, & \text{if } U_j \cap B_{\tilde{\Omega}}(\xi, R_j) \neq \emptyset, \\ 0, & \text{if } U_j \cap B_{\tilde{\Omega}}(\xi, R_j) = \emptyset. \end{cases}$$

Since the sets $U_j \cap B_{\tilde{\Omega}}(\xi, 2r)$ cover $B_{\tilde{\Omega}}(\xi, 2r)$ and $B_{\tilde{\Omega}}(\xi, r) \subset B_{\tilde{\Omega}}(\xi, R_k)$ for each k, to prove Lemma 4.6, it suffices to show that

$$\sup_{j \geq 0} d_j \leq A_2 < \infty$$

where A_2 is as in Lemma 4.6.

We proceed by iteration. Since $\omega_0 \leq 1$, we have by definition of U_0,

$$d_0 = \sup_{U_0 \cap B_{\tilde{\Omega}}(\xi, 2r)} \frac{\Psi(r)\omega_0(x)}{V(\xi, r)G'(x)} \leq \epsilon_1 e^2.$$

Let $j > 0$. For $x \in U_{j-1} \cap B_{\tilde{\Omega}}(\xi, R_{j-1})$, we have by definition of d_{j-1} that

$$\omega_0(x) \leq d_{j-1} \frac{V(\xi, r)}{\Psi(r)} G'(x).$$

Also, $\omega_0 \leq 1$. Therefore the maximum principle yields that, for $x \in V_j \cap B_{\tilde{\Omega}}(\xi, R_{j-1})$,

$$\omega_0(x) \leq \omega(x, V_j \cap \partial B_{\tilde{\Omega}}(\xi, R_{j-1}), V_j \cap B_{\tilde{\Omega}}(\xi, R_{j-1})) + d_{j-1} \frac{V(\xi, r)}{\Psi(r)} G'(x). \tag{24}$$

For $x \in V_j \cap B_{\tilde{\Omega}}(\xi, R_j)$, let $D = B(x, C_0^{-1}(R_{j-1} - R_j))$ and let D' be the connected component of $Y' \cap \Omega \cap D$ that contains x. Then by Lemma 3.4,

$$D' \subset B_{\tilde{\Omega}}(x, R_{j-1} - R_j) \subset B_{\tilde{\Omega}}(\xi, R_{j-1}),$$

hence $D' \cap \partial B_{\tilde{\Omega}}(\xi, R_{j-1}) = \emptyset$. Thus, the first term on the right hand side of (24) is not greater than

$$\begin{aligned}
&\omega\big(x, V_j \cap D' \cap \partial B(x, (R_{j-1} - R_j)/(2C_0)), V_j \cap D' \cap B(x, (R_{j-1} - R_j)/(2C_0))\big) \\
&\leq \exp\left(2 - \frac{a_1}{2C_0} \frac{R_{j-1} - R_j}{w_\eta(V_j \cap D')}\right) \\
&\leq \exp\left(2 - \frac{a_1}{2C_0} \frac{R_{j-1} - R_j}{w_\eta(V_j)}\right) \\
&\leq \exp\left(2 - \frac{a_1}{2C_0 A_4} \exp(2^j/\sigma) \frac{R_{j-1} - R_j}{r}\right) \\
&\leq \exp\big(2 - \epsilon_6 j^{-2} \exp(2^j/\sigma)\big)
\end{aligned}$$

by Lemma 4.5, monotonicity of $U \mapsto w_\eta(U)$ and (22). Here $\epsilon_6 = \frac{3a_1}{\pi^2 A_4 C_0}$. Moreover, by definition of U_j,

$$\epsilon_1 \frac{V(\xi,r)}{\Psi(r)} G'(x) \geq \exp(-2^{j+1})$$

for $x \in U_j$. Hence, for $x \in U_j \cap B_{\tilde{\Omega}}(\xi, R_j)$, (24) becomes

$$\begin{aligned}\omega_0(x) &\leq \exp\left(2 - \epsilon_6 j^{-2} \exp(2^j/\sigma)\right) + d_{j-1} \frac{V(\xi,r)}{\Psi(r)} G'(x) \\ &\leq \left(\epsilon_1 \exp\left(2 + 2^{j+1} - \epsilon_6 j^{-2} \exp(2^j/\sigma)\right) + d_{j-1}\right) \frac{V(\xi,r)}{\Psi(r)} G'(x).\end{aligned}$$

Dividing both sides by $\frac{V(\xi,r)}{\Psi(r)} G'(x)$ and taking the supremum over $x \in U_j \cap B_{\tilde{\Omega}}(\xi, R_j)$,

$$d_j \leq \epsilon_1 \exp\left(2 + 2^{j+1} - \epsilon_6 j^{-2} \exp(2^j/\sigma)\right) + d_{j-1},$$

and hence for every integer $i > 0$,

$$d_i \leq \epsilon_1 e^2 \left(1 + \sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{3a_1}{\pi^2 A_4 C_0} j^{-2} \exp(2^j/\sigma)\right)\right) = \epsilon_1 e^2 (1 + C) < \infty$$

by (23). □

Proof of 4.1 We follow [8][Theorem 4.5] and [1][Lemma 3]. Recall that $A_0 = A_3 + 7 = 2(12 + 12C_u) + 7$. Fix $\xi \in \partial\Omega$ with $R_\xi > 0$. Let $0 < r < \{R_{\xi'} : \xi' \in \partial\Omega \cap B_{\tilde{\Omega}}(\xi, 7R_\xi)\}$ and set $Y' = B_{\tilde{\Omega}}(\xi, A_0 r)$. Note that any two points in $B_{\tilde{\Omega}}(\xi, 12r)$ can be connected by a (c_u, C_U)-inner uniform path that stays in $B_{\tilde{\Omega}}(\xi, A_3 r/2)$.

Fix $x^* \in B_{\tilde{\Omega}}(\xi, r)$, $y^* \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$ such that $c_1 r \leq d(x^*, \partial\Omega) \leq r$ and $6c_0 r \leq d(y^*, \partial\Omega) \leq 6r$, for some constants $c_0, c_1 \in (0,1)$ depending on c_u and C_u. Existence of x^* and y^* follows from the inner uniformity of Ω. It suffices to show that for all $x \in B_{\tilde{\Omega}}(\xi, r)$ and $y \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$ we have

$$G_{Y'}(x,y) \asymp \frac{G_{Y'}(x^*, y)}{G_{Y'}(x^*, y^*)} G_{Y'}(x, y^*). \tag{25}$$

Fix $y \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$, and call u (v, respectively) the left(right)-hand side of (25), viewed as a function of x. Then u is positive and L-harmonic in $Y' \setminus \{y\}$, whereas v is positive and L-harmonic in $Y' \setminus \{y^*\}$. Both functions vanish quasi-everywhere on the boundary of Y'.

Since $y^* \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$ and $6c_0 r \leq d(y^*, \partial\Omega) \leq 6r$, it follows that the ball $B_{\tilde{\Omega}}(y^*, 3c_0 r)$ is contained in $B_{\tilde{\Omega}}(\xi, 9r) \setminus B_{\tilde{\Omega}}(\xi, 3r)$. Let $z \in \partial B_{\tilde{\Omega}}(y^*, c_0 r)$. By a repeated use of Harnack inequality (a finite number of times, depending only on c_u and C_u), one can compare the value of v at z and at x^*, so that by Lemma 3.8 (notice that $d(x^*, y) \geq c_1 r$) and the volume doubling property,

$$v(z) \leq C_1 v(x^*) = C_1 G_{Y'}(x^*, y) \leq C_1' \frac{\Psi(r)}{V(\xi, r)}.$$

Now, if $y \in B_{\tilde{\Omega}}(y^*, 2c_0 r)$, then by Lemma 3.8 (notice that $d_\Omega(z, y) \le 3r \le \frac{A_0 r}{6C_u}$ and $z, y \in B_{\tilde{\Omega}}(\xi, A_0 r/6)$) and the volume doubling property,

$$u(z) = G_{Y'}(z, y) \ge C_2 \frac{\Psi(r)}{V(\xi, r)},$$

so that we have $u(z) \ge C_3 v(z)$ in this case for some $C_3 > 0$. If instead $y \in \Omega \setminus B_{\tilde{\Omega}}(y^*, 2c_0 r)$, then we can connect z and x^* by a path of length comparable to r that stays away (at scale r) from both $\partial\Omega$ and the point y. Hence, the Harnack inequality implies that $u(z) \asymp u(x^*) = v(x^*) \asymp v(z)$ in this case. This shows that we always have

$$u(z) \ge \epsilon_3 v(z) \quad \forall z \in \partial B_\Omega(y^*, c_0 r).$$

By the maximum principle, we obtain

$$u \ge \epsilon_3 v \quad \text{ on } Y' \setminus B_{\tilde{\Omega}}(y^*, c_0 r).$$

Since $B_\Omega(\xi, r) \subset Y' \setminus B_{\tilde{\Omega}}(y^*, c_0 r)$, we have proved that $u \ge \epsilon_3 v$ on $B_{\tilde{\Omega}}(\xi, r)$, that is,

$$G_{Y'}(x, y) \ge \epsilon_3 \frac{G_{Y'}(x^*, y)}{G_{Y'}(x^*, y^*)} G_{Y'}(x, y^*) \tag{26}$$

for all $x \in B_{\tilde{\Omega}}(\xi, r)$ and $y \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$. This is one half of (25).

We now focus on the other half of (25), that is,

$$\epsilon_4 G_{Y'}(x, y) \le \frac{G_{Y'}(x^*, y)}{G_{Y'}(x^*, y^*)} G_{Y'}(x, y^*), \tag{27}$$

for all $x \in B_\Omega(\xi, r)$ and $y \in \Omega \cap \partial B_\Omega(\xi, 6r)$.

For $x \in B_\Omega(\xi, 2r)$ and $z \in B_{\tilde{\Omega}}(\xi, 9r) \setminus B_{\tilde{\Omega}}(\xi, 3r)$, Lemma 3.8 and the volume doubling condition yield

$$G_{Y'}(x, z) \le C_1 \frac{\Psi(r)}{V(\xi, r)}.$$

Regarding $G_{Y'}(x, z)$ as L-harmonic function of x, the maximum principle gives

$$G_{Y'}(\cdot, z) \le C_1 \frac{\Psi(r)}{V(\xi, r)} \omega(\cdot, \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 2r), B_{\tilde{\Omega}}(\xi, 2r)) \quad \text{ on } B_{\tilde{\Omega}}(\xi, 2r).$$

Using Lemma 4.6 (note that $A_0 > A_3$) and the Harnack inequality (to move from ξ_{16r} to y^*), we get for $x \in B_\Omega(\xi, r)$ and $z \in B_{\tilde{\Omega}}(\xi, 9r) \setminus B_{\tilde{\Omega}}(\xi, 3r)$, that

$$G_{Y'}(x, z) \le C_1 A_2 \frac{\Psi(r)}{V(\xi, r)} \frac{V(\xi, r)}{\Psi(r)} G_{Y'}(x, \xi_{16r}) \le C_4 G_{Y'}(x, y^*), \tag{28}$$

for some constant $C_4 > 0$. Fix $x \in B_\Omega(\xi, r)$ and $y \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$. If $d_\Omega(y, X \setminus \Omega) \ge c_0 r/2$, then $G_{Y'}(x, y) \asymp G_{Y'}(x, y^*)$ and $G_{Y'}(x^*, y) \asymp G_{Y'}(x^*, y^*)$ by the Harnack inequality, so that (27) follows. Hence we now assume that

$y \in \Omega \cap \partial B_{\tilde{\Omega}}(\xi, 6r)$ satisfies $d_\Omega(y, X \setminus \Omega) < c_0 r/2$. Let $\xi' \in \partial\Omega$ be a point such that $d_\Omega(y, \xi') < c_0 r/2$. It follows that $y \in B_{\tilde{\Omega}}(\xi', r)$. Also,

$$B_{\tilde{\Omega}}(\xi', 2r) \subset B_{\tilde{\Omega}}(y, 3r) \subset B_{\tilde{\Omega}}(\xi, 9r) \setminus B_{\tilde{\Omega}}(\xi, 3r).$$

We apply inequality (28) to get $G_{Y'}(x, z) \leq C_4 G_{Y'}(x, y^*)$ for any $z \in B_{\tilde{\Omega}}(\xi', 2r)$. Regarding $G_{Y'}(x, y)$ as L-harmonic function of y, we obtain

$$G_{Y'}(x, y) \leq C_4 G_{Y'}(x, y^*)\, \omega(y, \Omega \cap \partial B_{\tilde{\Omega}}(\xi', 2r), B_{\tilde{\Omega}}(\xi', 2r)). \tag{29}$$

Let us apply Lemma 4.6 with ξ replaced by ξ'. This yields

$$\begin{aligned} \omega(y, \Omega \cap \partial B_{\tilde{\Omega}}(\xi', 2r), B_{\tilde{\Omega}}(\xi', 2r)) &\leq A_2 \frac{V(\xi', r)}{\Psi(r)} G_{B_{\tilde{\Omega}}(\xi', C_0 A_3 r)}(y, \xi'_{16r}) \\ &\leq A_2' \frac{V(\xi, r)}{\Psi(r)} G_{Y'}(\xi'_{16r}, y), \end{aligned} \tag{30}$$

where $\xi'_{16r} \in \Omega$ is any point such that $d_\Omega(\xi'_{16r}, \xi') = 4r$ and $d(\xi'_{16r}, X \setminus \Omega) \geq 2c_u r$. Observe that we have used the volume doubling property as well as the set monotonicity of the Green function, and that $B_{\tilde{\Omega}}(\xi', A_3 r) \subset B_{\tilde{\Omega}}(\xi, A_0 r)$ because $A_0 = A_3 + 7$ and $d_\Omega(\xi, \xi') \leq 7r$. Now, (29) and (30) give

$$G_{Y'}(x, y) \leq C_5 \frac{V(\xi, r)}{\Psi(r)} G_{Y'}(\xi'_{16r}, y) G_{Y'}(x, y^*). \tag{31}$$

By construction, $d_\Omega(\xi'_{16r}, y) \geq d(\xi'_{16r}, \xi') - d_\Omega(\xi', y) \geq 2r$ and $d_\Omega(x^*, y) \geq d_\Omega(\xi, y) - d_\Omega(\xi, x^*) \geq 5r$. Using the inner uniformity of Ω, we find a chain of balls, each of radius $\asymp r$ and contained in $Y' \setminus \{y\}$, going from x^* to ξ'_{16r}, so that the length of the chain is uniformly bounded in terms of c_u, C_u. Applying the Harnack inequality repeatedly thus yields $G_{Y'}(\xi'_{16r}, y) \asymp G_{Y'}(x^*, y)$. As Lemma 3.8 gives $G_{Y'}(x^*, y^*) \asymp \Psi(r)/V(\xi, r)$, inequality (31) implies (27). This completes the proof. □

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